Journal of Statistics and Computer Science
Vol. 3, Issue 1, 2024, pp. 51-78
https://doi.org/10.47509/JSCS.2024.v03i01.04

Attrition coefficient estimations via differential equation systems, initial and terminal conditions, and nonlinear iterative equation system solutions

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Abstract

In battles with aimed fire, the attrition of a force can under simplified assumptions be shown to be proportional to the number of enemies. Lanchester models for aimed fire are differential equation systems that can be applied to describe the dynamics of such battles In order to determine the attrition coefficients and the complete dynamics of the battle in continuous time, the following procedure is introduced: First, the general solution of the Lanchester differential equation system, which is a homogenous second order differential equation system, is derived. The four parameters of the solution are determined. In these equations, the initial and terminal sizes of the two forces, are parameters. A 4-dimensional fix point iteration algorithm is developed and implemented as a computer code, that rapidly solves the nonlinear equation system. After 40 iterations, the absolute relative errors in all equations are smaller than 10^{-12} . Then, a discrete time version of the Lanchester differential equation system, with stochastic attrition coefficients, is defined as a difference equation system. The effects of increasing risk in the attrition coefficients, that determine how the time derivative of the size of force X is affected by the size of force Y, at different points in time, is analyzed. It is shown that the expected size of force X is a strictly convex function of the risk in the attrition coefficients. According to the Jensen's inequality, the expected size of force X at time t+2 is a strictly increasing function of the risk in the attrition coefficients at time t and t+1 for arbitrary values of t. In case the attrition coefficients in different periods are stochastic, and the system parameters are determined according to the suggested procedure, then the expected attrition coefficients obtain higher values than if the attrition coefficients would be constant over time. This can explain differences between attrition coefficient estimates based on different methods and coefficient risk assumptions.

Keywords: Lanchester equations, attrition parameters, differential equation system, numerical iteration.

1. Introduction

Competition can be observed in many different areas. In the domain of economics, we find competition between nations, in international trade theory, between companies, in market theory, and between individuals, in labor economics. Shatz (2020) gives a wide perspective on connected issues. Biological theory includes models of competition between different species, including many types of animals and plants. Compare the field covered by Iannelli and Pugliese (2014). Competition between nations and coalitions can also lead to wars and other conflicts. Relevant mathematical theories and examples are found in Washburn and Kress (2009). In all of these kinds of competition, we find several interesting and relevant scientific questions, such as: How do the different parties in the competition affect the other parties? How will the system develop over time? Can some actors influence these competitive situations and may optimal strategies be derived?

Article History
Received : 01 April 2024; Revised : 11 June 2024, Accepted : 21 June 2024; Published : 30 June 2024

To cite this paper

Peter Lohmander (2024). Attrition coefficient estimations via differential equation system. *Journal of Statistics and Computer Science*. 3(1), 51-78.

When scientific models are developed to describe, analyze, and manage the competition situations in economics, biology, and war science, it often turns out that the mathematical structure is very similar. In this study, we will focus on typical military problems. The general results and approaches can however be expected to be useful also in the fields of biology and economics. Wars are military conflicts, usually between nations. Sometimes, the participants belong to, or are cooperating with, other nations or coalitions. A recent study of how such wars can be modelled, and the strategies optimized, using optimal control theory, is Lohmander (2023). Key ingredients in that study are differential equations that show how the involved parties influence each other, via attrition warfare, and how the total war system can be controlled and optimized via external arms support. Wars can also be studied at lower levels of command and within more constrained geographical areas. Lohmander (2019a) and Lohmander (2019b) are two such examples.

In military operations research, the famous article by Lanchester (1916) is often used as a mathematical foundation. There, the general idea that the sizes of two opposing forces, X and Y, change over time, according to principles expressed as two differential equations. One of these differential equation systems, which has often been found to fit empirical time series data, from real battles, very well, states that the time derivative of the size of force X, is negative and proportional to the size of force Y. Furthermore, the time derivative of the size of force Y, is negative and proportional to the size of force X. In battles with aimed fire, the attrition of a force can under simplified assumptions be shown to be proportional to the number of enemies. Lanchester models for aimed fire are differential equation systems that can be applied to describe the dynamics of such battles. Estimations of attrition coefficients, the force reductions per time unit, per unit of the enemy force, have been reported in the literature, based on time series data from historical battles. Engel (1954), Bracken (1995), Tam (1998), Hung et al (2005) and Stymfal (2022) include such applications and estimations of the Lanchester models based on real military time series from different battles. Braun (1993) describes some of the applied differential equations and approaches.

Relevant empirical data would ideally contain complete time series of the numbers of units of both forces. Sometimes, the time series are incomplete, and only the time series of one force is known. In some cases, the time series of one force is completely known, but only the initial and the final sizes of the enemy force are known. In earlier research, estimations of attrition coefficients have sometimes been made in discrete time, based on the observed time series data of one force, X, and the assumed and calculated time path of the size of the other force, Y. Such estimations have been made in several steps. This study will first investigate the general attrition coefficient estimation problem in continuous time. The relevant general differential equation system is specified and analytically solved and the parameters are numerically determined via numerical fix point iteration. In this process, the initial and terminal sizes of the involved forces are parameters of the boundary conditions.

Often, deterministic models are approximations of a reality that is not perfectly predictable. Of course, this is true also in the present area of analysis. Rothschild and Stiglitz ((1970), (1971)) define risk, and increasing risk, in mathematically convenient ways, which makes it possible to study how stochastic parameter variations affect variables, systems, and optimal decisions. Lohmander (1986) and (1988) combines and applies the risk definitions of Rothschild and Stiglitz ((1970), (1971)) with the famous Jensen's inequality, biological production functions and price series of natural resources, via analytical stochastic dynamic programming, to show how increasing risk in market prices and growth processes dynamically affect optimal decision in biological production. In a similar way, stochastic parameters should be expected to influence the outcomes of dynamic competition, battles, and wars. For this reason, a discrete time difference equation system approximation, of the Lanchester differential equation system, is developed. With the created difference equation system model, the dynamic effects of stochastic attrition parameters are analyzed. The obtained results are reported in this paper.

The battle of Iwo Jima, an island in the Pacific Ocean, occurred between United States and Japan during the second World War, from February 19 until March 26, 1945. Engel (1954), Braun (1993), Washburn and Kress (2009), and Stymfal (2022), have all described and analyzed this battle. The original time series data and estimations from Engel (1954) have over time been marginally updated. In Figure 1, we see the latest version of the time series data of the force of USA, denoted X, on the island Iwo Jima, from the start of the battle, Day 0, until the end of the battle, Day 36. Figure 2 illustrates how the US force changed, per day, during the battle. Day 3 and Day 6, more troops from USA landed on the island. Note that the time series of the size of the Japanese force is not available. The initial and terminal sizes of the Japanese force have approximate values in all reported studies. A discussion of these approximations is found in Stymfal (2022).

In this paper, the general methodology is of key interest. The analysis does not introduce special assumptions to handle reinforcements. For this reason, the time series is analyzed only during the time interval without reinforcements. Figure 3 shows the changes of the US force from Day 6 until the end of the battle. The force difference during Day t, in Figure 3, is defined as the size of the force during Day t minus the size of the force during Day t-1. In the first analyses in this paper, we make the following assumptions, based on the data found in Stymfal (2022): We study the battle in continuous time, during the time interval from 0 to T, where T is 30. The time units are days, representing 24 hours each. In the present analysis, this time interval represents Day 6 until Day 36, as reported in Figure 1. At time 0, the size of the force from USA is 66 150 troops and Japan, according to a graph in Stymfal (2022), has 18 000 troops. At time T, USA has 52 135 troops and the size of the force from Japan, in Case 0 in this study, is 200 troops. (The terminal size of the Japanese force is not exactly known, and assumed to be different in the three different cases under analysis.)

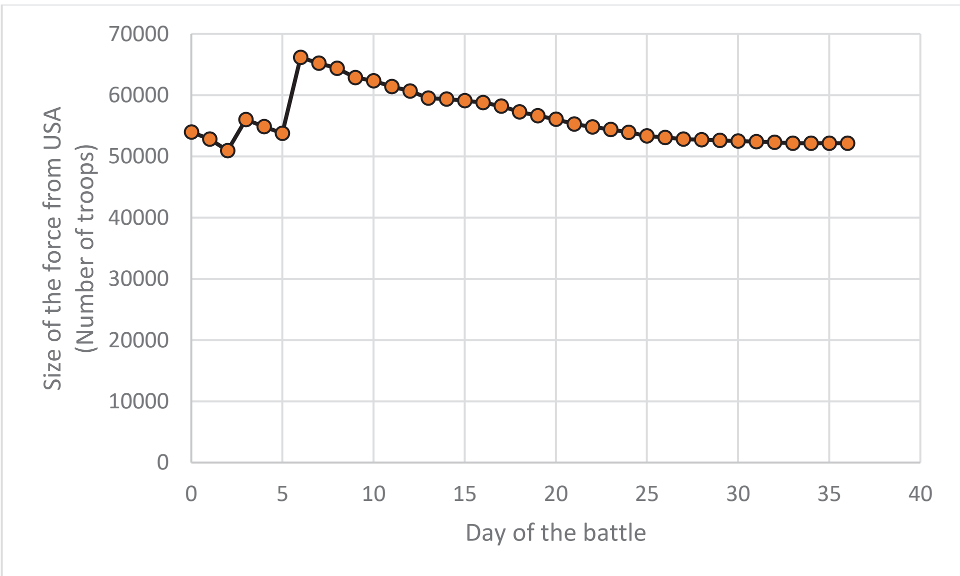

Figure 1. The dynamically changing size of force X, the force from USA, in the battle of Iwo Jima. Source: Data reported in Stymfal (2022), based on Engel (1954).

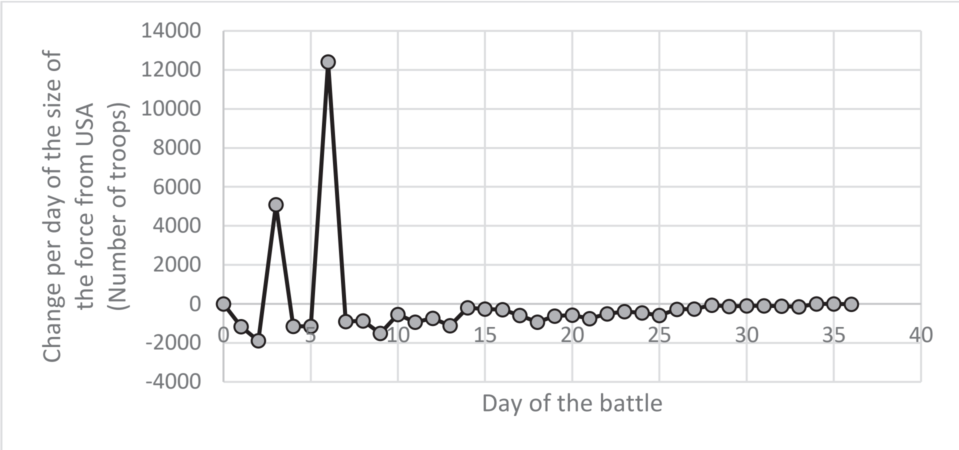

Figure 2. The change per day, the difference, of the size of force X, the force from USA, in the battle of Iwo Jima. Source: Derivations based on the data reported in Stymfal (2022), based on Engel (1954).

In Figure 2., the graph shows the differences from Day 0 until the end of the battle. The force difference during Day t, is defined as the size of the force during Day t minus the size of the force during Day t-1. (In the graph, the size of the force in Day -1 was assumed to be identical to the size of the force in Day 0.) Reinforcements took place Day 3 and Day 6.

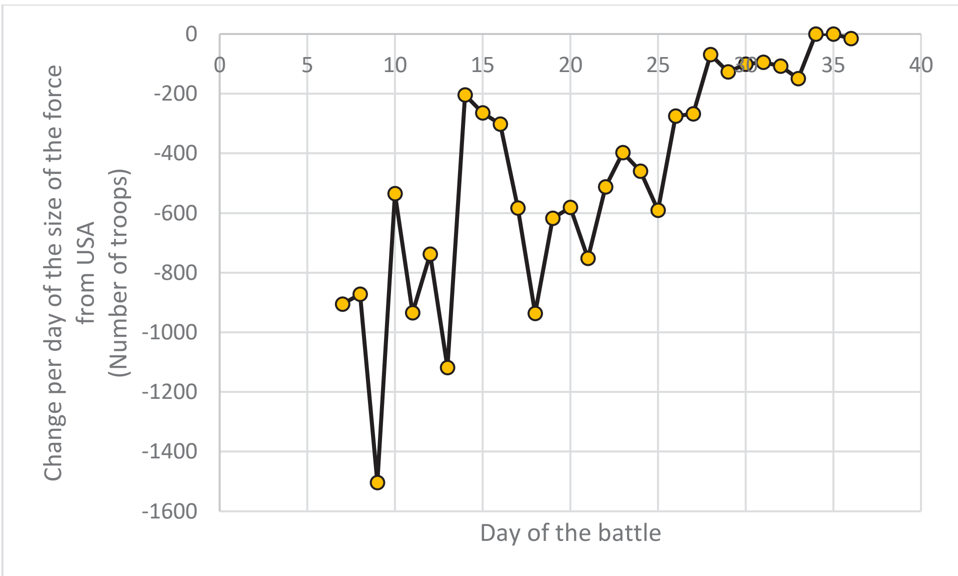

Figure 3. The change per day, the difference, of the size of force X, the force from USA, in the battle of Iwo Jima. Source: Derivations are based on the data reported in Stymfal (2022), based on Engel (1954).

In Figure 3., the graph shows the differences from Day 7 until the end of the battle. The force difference during Day t, is defined as the size of the force during Day t minus the size of the force during Day t-1. Hence, the graph is based on data covering the size of the force from Day 6 until Day 36.

2. Materials and Methods

Briefing on this section:

In order to determine the attrition coefficients and the complete dynamics of the battle in continuous time, the following procedure is used: First, the general solution to the Lanchester differential equation system, which is a homogenous second order differential equation system, is derived. This is a 2-dimensional Two Point Boundary Value Problem. Four parameters are determined, via a nonlinear simultaneous equation system with four equations. In these equations, the initial and terminal sizes of the two forces, are parameters. A 4-dimensional fix point iteration algorithm is developed and implemented as a computer code, that rapidly solves the nonlinear equation system. After 40 iterations, the absolute relative errors in all equations are less than 10^{-12} *.*

2.1. The differential equation system

We study the differential equation system (1). There we see how the state of the system, (x, y), representing the sizes of the two opposing forces, changes over time, $t, 0 \le t \le T \ll \infty$. The two parameters, (a,b), are denoted attrition coefficients. Newtonian notation, with time derivatives marked by dots, is used.

$$\begin{cases} \dot{x} = -ay & (1.a) \\ \dot{y} = -bx & (1.b) \end{cases} \qquad a > 0, b > 0 \tag{1}$$

From (1.a), we get (2).

$$y = -a^{-1}\dot{x} \tag{2}$$

Differentiation of (2) with respect to time, gives (3).

$$\dot{y} = -a^{-1}\ddot{x} \tag{3}$$

(3) and (1.b) give (4), which can be rewritten as (5) and (6), which is a homogenous second order differential equation.

$$-a^{-1}\ddot{x} = -bx \tag{4}$$

$$a^{-1}\ddot{x} - bx = 0 \tag{5}$$

$$\ddot{x} - abx = 0 \tag{6}$$

Let us assume that the functional form (7) is relevant. The parameters (m, λ) are assumed to be strictly different from zero.

$$x(t) = me^{\lambda t}, \quad m \neq 0, \lambda \neq 0, 0 \leq t \leq T \ll \infty \tag{7}$$

Then, the following procedure can be used to determine the state variable as an explicit function of time. Equations (6) and (7) give (8).

$$\lambda^2 me^{\lambda t} - abme^{\lambda t} = 0 \tag{8}$$

Equation (8) can be simplified to (9).

$$\left(\lambda^2 - ab\right)me^{\lambda t} = 0 \tag{9}$$

Equations (7) and (9) imply (10).

$$\lambda^2 - ab = 0 \tag{10}$$

From the quadratic equation (10), we obtain the solution (11).

$$\lambda = \pm\sqrt{ab} \tag{11}$$

Let r be defined according to (12).

$$r = \sqrt{ab} \tag{12}$$

Clearly, two solutions exist.

$$\lambda_1 = -r \tag{13}$$

$$\lambda_2 = r \tag{14}$$

Observation:

$a > 0 \wedge b > 0$, as we see in equation (1), which means that there are two real roots. These roots have different values. Hence, the general solution of the differential equation is:

$$x(t) = m_1 e^{-rt} + m_2 e^{rt} \tag{15}$$

Furthermore, from (2) we already know that: $y = -a^{-1}\dot{x}$

As a result, we get (16).

$$y(t) = -a^{-1}\left(-rm_1 e^{-rt} + rm_2 e^{rt}\right) \tag{16}$$

The expression (16) may be rewritten as (17).

$$y(t) = \frac{r}{a} m_1 e^{-rt} - \frac{r}{a} m_2 e^{rt} \tag{17}$$

Hence, the solution to the differential equation system (1) is given in (18).

$$\begin{cases} x(t) = & m_1 e^{-rt} + m_2 e^{rt} \\ y(t) = & \frac{r}{a} m_1 e^{-rt} - \frac{r}{a} m_2 e^{rt} \end{cases} \tag{18}$$

2.2. Determination of the parameters

To determine the time path $\left(x(t), y(t)\right)$ we need to know the four parameters $\left(m_1, m_2, a, r\right)$.

We may use four boundary conditions to determine these parameters. We already know the initial and terminal conditions of the system, namely $\left(x_0, y_0\right)$ and $\left(x_T, y_T\right)$.

From equation (18), the initial conditions (19) and (20) follow:

$$x(0) = m_1 + m_2 = x_0 \tag{19}$$

$$y(0) = \frac{r}{a} m_1 - \frac{r}{a} m_2 = y_0 \tag{20}$$

The terminal conditions, (21) and (22), are also derived from equation (18):

$$x(T) = m_1 e^{-rT} + m_2 e^{rT} = x_T \tag{21}$$

$$y(T) = \frac{r}{a} m_1 e^{-rT} - \frac{r}{a} m_2 e^{rT} = y_T \tag{22}$$

We may now determine the values of the parameters $\left(m_1, m_2, a, r\right)$. The nonlinear simultaneous equation system (23) should be solved. We assume that a feasible solution exists and that this solution is unique.

$$\begin{cases} m_1 + m_2 & = x_0 & (23.a) \\ m_1 e^{-rT} + m_2 e^{rT} & = x_T & (23.b) \\ \frac{r}{a} m_1 - \frac{r}{a} m_2 & = y_0 & (23.c) \\ \frac{r}{a} m_1 e^{-rT} - \frac{r}{a} m_2 e^{rT} & = y_T & (23.d) \end{cases} \tag{23}$$

The solution of the simultaneous equation system can be found via an iteration algorithm, which is developed here:

The initial guesses of the values of the parameters are given in (24).

$$\left(m_1, m_2, a, r\right) = \left(m_1^0, m_2^0, a^0, r^0\right) \tag{24}$$

The values of $\left(m_1, m_2, a, r\right)$ are sequentially updated. The iteration number is $i, i \in \{0,1,...,I\}$. The value of a parameter, η, after i iteration steps, is denoted η^i.

$$\left(Eq.23.a\right) \Rightarrow \left(m_1^{i+1} = x_0 - m_2^i\right) \tag{25}$$

$$\left(Eq.23.b\right) \Rightarrow \left(m_2^{i+1} = x_T e^{-r^i T} - m_1^{i+1} e^{-2r^i T}\right) \tag{26}$$

$$\left(Eq.23.c\right) \Rightarrow \left(a^{i+1} = \frac{r^i (m_1^{i+1} - m_2^{i+1})}{y_0}\right) \tag{27}$$

$$\left(Eq.23.d\right) \Rightarrow \left(\frac{r^i}{a^{i+1}} m_2^{i+1} e^{r^{i+1}T} = \frac{r^i}{a^{i+1}} m_1^{i+1} e^{-r^i T} - y_T\right) \tag{28}$$

From (28), we get (29), (30) and (31).

$$e^{r^{i+1}T} = \frac{\left(\frac{r^i}{a^{i+1}} m_1^{i+1} e^{-r^i T} - y_T\right)}{\frac{r^i}{a^{i+1}} m_2^{i+1}} \tag{29}$$

$$r^{i+1}T = LN\left(\frac{\left(\frac{r^i}{a^{i+1}} m_1^{i+1} e^{-r^i T} - y_T\right)}{\frac{r^i}{a^{i+1}} m_2^{i+1}}\right) \tag{30}$$

$$r^{i+1} = \frac{LN\left(\frac{\left(\frac{r^i}{a^{i+1}} m_1^{i+1} e^{-r^i T} - y_T\right)}{\frac{r^i}{a^{i+1}} m_2^{i+1}}\right)}{T} \tag{31}$$

However, even if (31) is mathematically correct, it turns out that the solution to the equation system (23) sometimes diverges if equation (31) is used directly. This means that if the initial parameter guesses (24) are not quite correct, the iteration method will not make the solution approach the correct solution. Fortunately, as will be shown, it is very easy to obtain convergence in the algorithm. In the modified algorithm, the value of the parameter r changes less rapidly than if equation (31) would be used directly. The absolute change of r in the adjusted algorithm, is smaller than according to (31), but the change of r is proportional to the change that would take place if (31) would be directly applied. In equation (32), the adjustment speed parameter h is introduced.

$$r^{i+1} = r^i + h\left(\frac{LN\left(\frac{\left(e^{-r^iT} \frac{r^i}{a^{i+1}} m_1^{i+1} - y_T \right)}{\left(\frac{r^i}{a^{i+1}} m_2^{i+1} \right)} \right)}{T} - r^i \right) \tag{32}$$

If $h=1$, then equation (32) corresponds exactly to (31), and the solution has been observed to diverge from the equilibrium. In the tested applications, the algorithm converges rapidly if we select the adjustment speed parameter value $h^* = 0.3$.

2.3 Summary of the algorithm

The initial values of the parameters are introduced in equation (33).

$$\left(m_1, m_2, a, r\right) = \left(m_1^0, m_2^0, a^0, r^0\right) \tag{33}$$

The values of $\left(m_1, m_2, a, r\right)$ are sequentially updated. The iteration number is $i, i \in \{0,1,...,I\}$.

$$m_1^{i+1} = x_0 - m_2^i \tag{34}$$

$$m_2^{i+1} = x_T e^{-r^iT} - m_1^{i+1} e^{-2r^iT} \tag{35}$$

$$a^{i+1} = \frac{r^i (m_1^{i+1} - m_2^{i+1})}{y_0} \tag{36}$$

$$r^{i+1} = r^i + h\left(\frac{LN\left(\frac{\left(e^{-r^i T} \frac{r^i}{a^{i+1}} m_1^{i+1} - y_T \right)}{\left(\frac{r^i}{a^{i+1}} m_2^{i+1} \right)} \right)}{T} - r^i \right), \quad h = 0.3 \tag{37}$$

In case the solution does not converge in some other application, it is suggested that the adjustment speed parameter h is reduced to some value such that $0 < h < 0.3$. The iteration algorithm in equations (33) to (37), is implemented in the computer code in the numerical appendix, and used to solve the coefficient estimation problems with empirical data.

2.4 Parameter estimation based on discrete time and stochastic outcomes

Briefing on this section:

Now, a discrete time version of the Lanchester simultaneous differential equation system, with stochastic attrition coefficients, is defined as a difference equation system. Stochastic variables are added to the expected attrition coefficients in different time periods, keeping the expected value of the attrition coefficients constant. The effect of increasing risk in the attrition coefficients that determine how the time derivative of force X is affected by force Y at different points in time, is analyzed. It is shown that the expected size of the force X at time t+2, is a strictly convex function of the risk in the attrition coefficients at times t and t+1. Hence, according to the Jensen's inequality, the expected size of the force X at time t+2, is a strictly increasing function of the risk in the attrition coefficients. Comparative statics analysis shows that, in case the attrition coefficients in different periods are stochastic, but the system parameters are determined under deterministic assumptions, then the estimated attrition coefficients obtain higher values than if the attrition coefficients would really be constant over time. This can partly explain differences between attrition coefficient estimates based on different methods and risk assumptions.

2.5 Development of the stochastic difference model

The initial conditions (x_0, y_0) are known. s is a stochastic variable with expected value $E(s) = 0$, the variance δ_s^2 and the standard deviation δ_s. If the stochastic variable s increases in one period, it decreases in another period, so that the expected value of the attrition coefficients in different periods, is held constant. The attrition coefficients in different periods are given period indices and are assumed to be strictly positive. The state (x_t, y_t) variables are also assumed to be strictly positive, as seen in equation (38). We focus on risk in the periods 0 and 1, and make sure that the expected value of the attrition coefficients in these periods is held constant. Compare equation (39).

$$a_t > 0 \wedge b_t > 0 \wedge x_t > 0 \wedge y_t > 0, \quad t \in \{0,1,...,T\} \tag{38}$$

$$a = \frac{a_0 + a_1}{2} \tag{39}$$

We can express the time dependent attrition coefficients as (40) and (41).

$$a_0 = a + s \tag{40}$$

$$a_1 = a - s \tag{41}$$

The coordinates at time t are (x_t, y_t). These are recursively determined in (42) to (45).

$$\Delta x_1 = x_1 - x_0 = -a_0 y_0 \tag{42}$$

$$\Delta y_1 = y_1 - y_0 = -b_1 x_1 \tag{43}$$

$$\Delta x_2 = x_2 - x_1 = -a_1 y_1 \tag{44}$$

$$\Delta y_2 = y_2 - y_1 = -b_2 x_2 \tag{45}$$

The recursion (42) to (45) can be described as (46) to (49).

$$x_1 = x_0 - a_0 y_0 \tag{46}$$

$$y_1 = y_0 - b_1 x_1 \tag{47}$$

$$x_2 = x_1 - a_1 y_1 \tag{48}$$

$$y_2 = y_1 - b_2 x_2 \tag{49}$$

Now, we can determine how x_2 is affected by changing properties of the stochastic variable s.

From (46) and (48), we get (50). Via the earlier equations, (50) is further developed to (51).

$$x_2 = \left(x_0 - a_0 y_0\right) - a_1 y_1 \tag{50}$$

$$x_2 = \left(x_0 - (a + s) y_0\right) - (a - s)\left(y_0 - b_1\left(x_0 - a_0 y_0\right)\right) \tag{51}$$

$$x_2 = x_0 - (a+s)y_0 - (a-s)y_0 + (a-s)b_1\left(x_0 - (a+s)y_0\right) \tag{52}$$

$$x_2 = x_0 - 2ay_0 + (a-s)b_1x_0 - (a-s)(a+s)b_1y_0 \tag{53}$$

$$x_2 = \left(1 + (a-s)b_1\right)x_0 - \left(2a + \left(a^2 - s^2\right)b_1\right)y_0 \tag{54}$$

$$\frac{dx_2}{ds} = -b_1x_0 + 2b_1y_0s \tag{55}$$

$$\frac{d^2x_2}{ds^2} = 2b_1y_0 > 0 \tag{56}$$

Observation:

x_2 may be regarded as a function of many parameters, including s. Compare equation (54). In equations (57) and (58), we simplify notation and write $x_2(s)$. According to equation (56), x_2 is a strictly convex function of s. From the Jensen's inequality (Jensen (1906)), we get the equations (57) and (58).

$$E\left(x_2(s)\right) > x_2\left(E(s)\right), \quad if\, \delta_s^{\,2} > 0 \tag{57}$$

$$E\left(x_2(s)\right) = x_2\left(E(s)\right), \quad if\, \delta_s^{\,2} = 0 \tag{58}$$

2.6 Terminal condition as expected value

In a 2- period problem, we have the terminal condition found in equation (59). The expected value of x_2 is written as a function of (a,s), where s is a function of the standard deviation of s, δ_s.

$$E\left(x_2\left(a, s(\delta_s)\right)\right) = x_T \tag{59}$$

We are interested to see how the estimated a should be adjusted in case we know that δ_s increases, and we simultaneously want to make sure that the terminal condition (59) is satisfied.

Total differentiation gives equation (60). Clearly, as we see in equation (61), we cannot change the already known terminal value of the state variable.

$$\frac{dE(x_2)}{da}da + \frac{dE(x_2)}{d\delta_s}d\delta_s - dx_T = 0 \tag{60}$$

$$dx_T = 0 \tag{61}$$

Equations (60) and (61) lead to (62), which can be rewritten as (63).

$$\frac{dE(x_2)}{da}da + \frac{dE(x_2)}{d\delta_s}d\delta_s = 0 \tag{62}$$

$$\frac{dE(x_2)}{da}da = -\frac{dE(x_2)}{d\delta_s}d\delta_s \tag{63}$$

The derivative of the parameter a , the estimated expected attrition coefficient, with respect to the standard deviation of the attrition coefficient, δ_s , is found in equation (64).

$$\frac{da}{d\delta_s} = \frac{-\left(\frac{dE(x_2)}{d\delta_s}\right)}{\left(\frac{dE(x_2)}{da}\right)} \tag{64}$$

In order to determine the sign of the derivative in equation (64), must know the sign of the derivative of x_2 with respect to a , which is found in (65). Equation (65) can be reformulated to (66) and (67).

$$\frac{dx_2}{da} = -2y_0 + b_1x_0 - 2b_1y_0a \tag{65}$$

$$\frac{dx_2}{da} = -2y_0(1 + b_1a) + b_1x_0 \tag{66}$$

$$\frac{dx_2}{da} = b_1x_0\left(-2\frac{(1+b_1a)}{b_1}\frac{y_0}{x_0} + 1\right) \tag{67}$$

Equation (68) shows a combination of three different assumptions, which makes sure that the sign of the derivative of x_2 with respect to a is strictly negative. The first listed assumption follows from the earlier assumptions in this paper. The second assumption is satisfied in case $b_1 < 0.1$, which is normal in most battles. (Compare the attrition coefficient values is Table 5.) The third assumption is a constraint on the ratio between the sizes of the initial forces: The initial size of force Y is at least 5% of the initial size of force X. That assumption is probably relevant in almost all real battles. Compare the initial force sizes reported in Table 1. In case the assumptions in (68) are true, then equation (69) follows.

$$(b_1x_0 > 0) \wedge \left(\frac{(1+b_1a)}{b_1} > 10\right) \wedge \left(\frac{1}{20} \le \frac{y_0}{x_0}\right) \Rightarrow \frac{dx_2}{da} < 0 \tag{68}$$

$$\left(\frac{dE(x_2)}{d\delta_s} > 0 \wedge \frac{dE(x_2)}{da} < 0\right) \Rightarrow \frac{da}{d\delta_s} = \frac{-\left(\frac{dE(x_2)}{d\delta_s}\right)}{\left(\frac{dE(x_2)}{da}\right)} > 0 \tag{69}$$

Some interpretations of equation (69) and the earlier assumptions are the following: We are interested to see how the estimated expected attrition coefficient a should be adjusted in case δ_s changes. Simultaneously, we want to make sure that the terminal condition (59) is satisfied. The estimated expected attrition coefficient a is a strictly increasing function of δ_s. In other words; If the attrition coefficients contain more stochastic variation, and the terminal size of force X is constant, then the estimated value of the expected attrition coefficient increases.

2.7 Generalization

In case the reader prefers a more general version of the theory developed in (68) and (69), we may study equation (70). Equations (64) and (67) imply equation (70). The first assumption written in equation (70) follows from the earlier assumptions in this paper. The second assumption is that the ratio $\frac{y_0}{x_0}$ exceeds a particular value, determined by the parameters (a, b_1). In case $b_1 = 0.1$, $\frac{y_0}{x_0}$ should exceed 0.05, to satisfy the constraint, for all $a \geq 0$. In case $b_1 = 0.2$, $\frac{y_0}{x_0}$ should exceed 0.10, to satisfy the constraint, for all $a \geq 0$. If the constraint on the initial force ratios is satisfied, then equation (69) is satisfied, which is also clear from equation (70). In other words; If the attrition coefficients contain more stochastic variation, and the terminal size of force X is constant, then the estimated value of the expected attrition coefficient increases.

$$(b_1 x_0 > 0) \wedge \left(\frac{y_0}{x_0} > \frac{b_1}{2(1+b_1 a)}\right) \Rightarrow \left(\frac{dx_2}{da} < 0\right) \Rightarrow \left(\frac{da}{d\delta_s} = \frac{-\left(\frac{dE(x_2)}{d\delta_s}\right)}{\left(\frac{dE(x_2)}{da}\right)} > 0\right) \tag{70}$$

3. Results

The general theory and the numerical iteration method developed in this paper are used to determine the attrition coefficients, and the dynamic developments of the forces from USA (= X), and Japan (=Y), during the battle of Iwo Jima, from Day 6 (t = 0), when all US troops had arrived to the island, until Day 36 (t = T = 30), the end of the battle.

Table 1 shows the initial and terminal sizes of the force from USA, X, based on the data from the empirical appendix, and the initial and terminal sizes of the Japanese force, Y, according to estimates from Stymfal (2022), also denoted "Case 0" of this study. The terminal size of the Japanese force is not known with certainty. Therefore, in this study, all results are also determined for $Y_T = 100$ (Case 1) and for $Y_T = 300$ (Case 2). Tables 1, 2, 3 and 4 are produced by the computer code, which is found in the numerical appendix. Note that the initial values of the estimated parameters differ considerably from the estimated parameters, found in table 3. Several combinations of initial parameter values are tested.

It is found that the iteration algorithm converges and is reliable. The results, the estimates of the parameters reported in Table 3, are not affected by the alternative initial parameter values.

Table 1. Parameters used in Case 0.

```
Initial and terminal conditions:

      x0 =  66150
      y0 =  18000
      xT =  52135
      yT =  200

  Other parameters:

      T =  30
      h =  .3

  Initial values of estimated parameters:

      a_0  =  .02
      b_0  =  .02
      r_0  =  .02
      m1_0 =  1
      m2_0 =  1
```

Table 2 shows how the numerical solution of the coefficient estimation algorithm converges. The convergence table is automatically generated by the computer code. The table shows how the relative errors in the four equations develop, during the numerical iteration. The first row corresponds to iteration 1, and the final row corresponds to iteration 40. After 40 iterations, the absolute relative errors are less than 10^{-12} in all the four equations.

Table 2. Iterative error reductions.

x0Err	y0Err	xTErr	yTErr
0.131331802328	0.406077556595	−0.066792339286	17.177549422846
0.046942030019	−0.044271399178	−0.001238733534	−2.424737248853
0.009794740047	−0.058589139025	−0.002141770808	−3.292782037948
0.002936436656	−0.035338780938	−0.000593458710	−1.960769298557
0.000903806604	−0.017852318448	−0.000083122461	−0.980041705441
0.000246411311	−0.008169564742	0.000011244902	−0.445823156844
0.000054828088	−0.003498047061	0.000014481506	−0.190341865254
0.000007624910	−0.001430777621	0.000007633629	−0.077753827990
−0.000001114673	−0.000567126973	0.000003306407	−0.030803137648
−0.000001551748	−0.000220022137	0.000001326350	−0.011947749717
−0.000000890008	−0.000084116219	0.000000513592	−0.004567327759
−0.000000414667	−0.000031836318	0.000000195333	−0.001728585582
−0.000000176158	−0.000011966158	0.000000073554	−0.000649706634
−0.000000071181	−0.000004476115	0.000000027533	−0.000243031058
−0.000000027914	−0.000001668771	0.000000010268	−0.000090605912
−0.000000010741	−0.000000620696	0.000000003819	−0.000033700634
−0.000000004082	−0.000000230489	0.000000001418	−0.000012514378
−0.000000001538	−0.000000085491	0.000000000526	−0.000004641745
−0.000000000577	−0.000000031684	0.000000000195	−0.000001720290
−0.000000000215	−0.000000011736	0.000000000072	−0.000000637198
−0.000000000080	−0.000000004345	0.000000000027	−0.000000235924
−0.000000000030	−0.000000001608	0.000000000010	−0.000000087327
−0.000000000011	−0.000000000595	0.000000000004	−0.000000032317
−0.000000000004	−0.000000000220	0.000000000001	−0.000000011958
−0.000000000002	−0.000000000081	0.000000000001	−0.000000004424
−0.000000000001	−0.000000000030	0.000000000000	−0.000000001637
−0.000000000000	−0.000000000011	0.000000000000	−0.000000000605
−0.000000000000	−0.000000000004	0.000000000000	−0.000000000224
−0.000000000000	−0.000000000002	0.000000000000	−0.000000000083
−0.000000000000	−0.000000000001	0.000000000000	−0.000000000031
−0.000000000000	−0.000000000000	0.000000000000	−0.000000000011
−0.000000000000	−0.000000000000	0.000000000000	−0.000000000004
−0.000000000000	−0.000000000000	0.000000000000	−0.000000000002
−0.000000000000	−0.000000000000	0.000000000000	−0.000000000001
0.000000000000	−0.000000000000	0.000000000000	−0.000000000000
0.000000000000	−0.000000000000	0.000000000000	−0.000000000000
0.000000000000	−0.000000000000	0.000000000000	−0.000000000000
0.000000000000	0.000000000000	0.000000000000	0.000000000000
0.000000000000	0.000000000000	0.000000000000	0.000000000000
0.000000000000	0.000000000000	0.000000000000	0.000000000000

The estimated parameter values of Case 0 are found in Table 3. The computer code also automatically derives and prints the estimated force equations, as seen in Table 4.

Table 3. Estimated parameter values of Case 0.

```
a  =  5.347041320955464D-02
b  =  .0104491786465644
r  =  2.363729891362914D-02
m1 =  53434.08250957198
m2 =  12715.91749042803
```

Table 4. Estimated force equations of Case 0.

```
x(t) =  53434.083 * EXP(-0.02363730 * t )  +  12715.917 * EXP( 0.02363730 * t )

y(t) =  23621.238 * EXP(-0.02363730 * t )  -   5621.238 * EXP( 0.02363730 * t )
```

The estimated force size function based on the Case 0 assumptions, is denoted F0_USA. The time series of the force from USA, found in the empirical appendix, fits the estimated function, F0_USA, very well. This is seen in Figure 4.

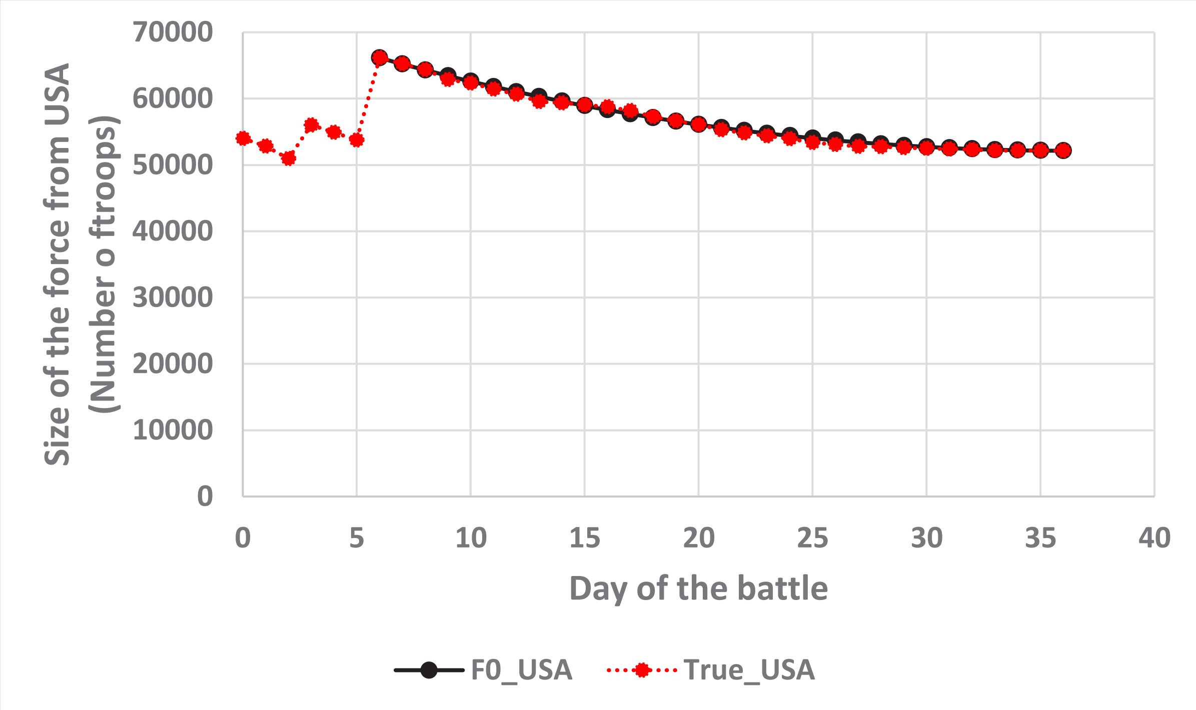

Figure 4. Size of the force from USA, according to the true (empirical) time series and according to the model version F0_USA.

The dynamic changes of the force from USA and Japan, according to the estimated functions, based on Case 0, denoted F0_USA and F0_Japan, are shown in Figure 5.

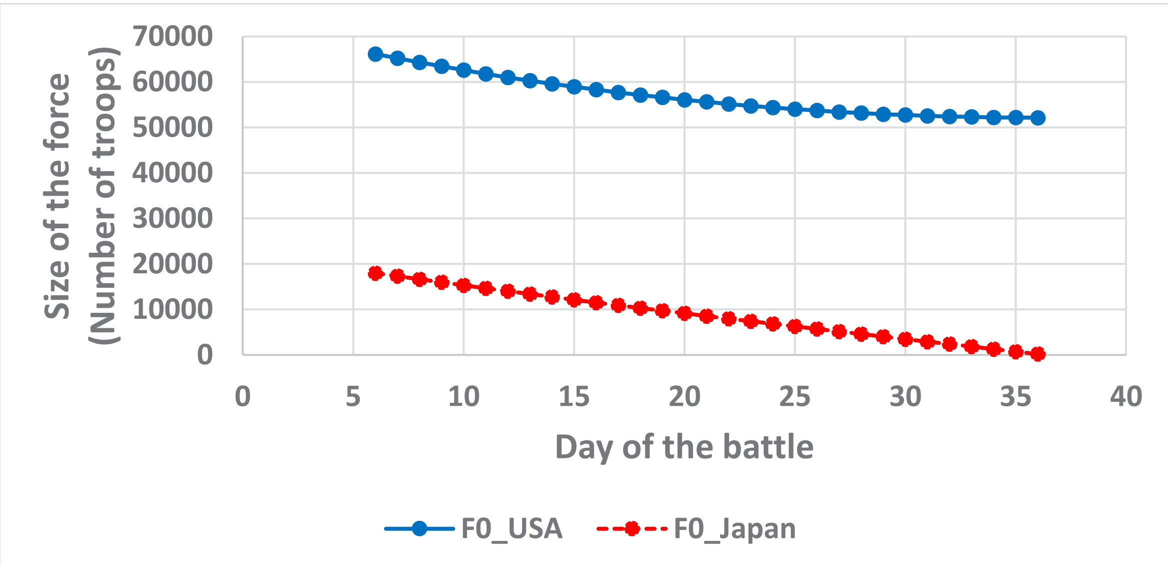

Figure 5. Sizes of the forces from USA and Japan, according to the models F0_USA and F0_Japan.

Figure 6 shows that the different tested assumptions concerning the terminal size of the force from Japan, Y_T, do not dramatically influence the dynamic force changes. The differences between different assumptions are almost not visible in the reported scale. In Case 0, Y_T =200, in Case 1, Y_T = 100, and in Case 2, Y_T =300. In Case 0, we have F0_USA and F0_Japan. In Case 1, we get F1_USA and F1_Japan. Case 2 gives F2_USA and F2_Japan. The Figures 7, 8 and 9 show the differences in higher resolution.

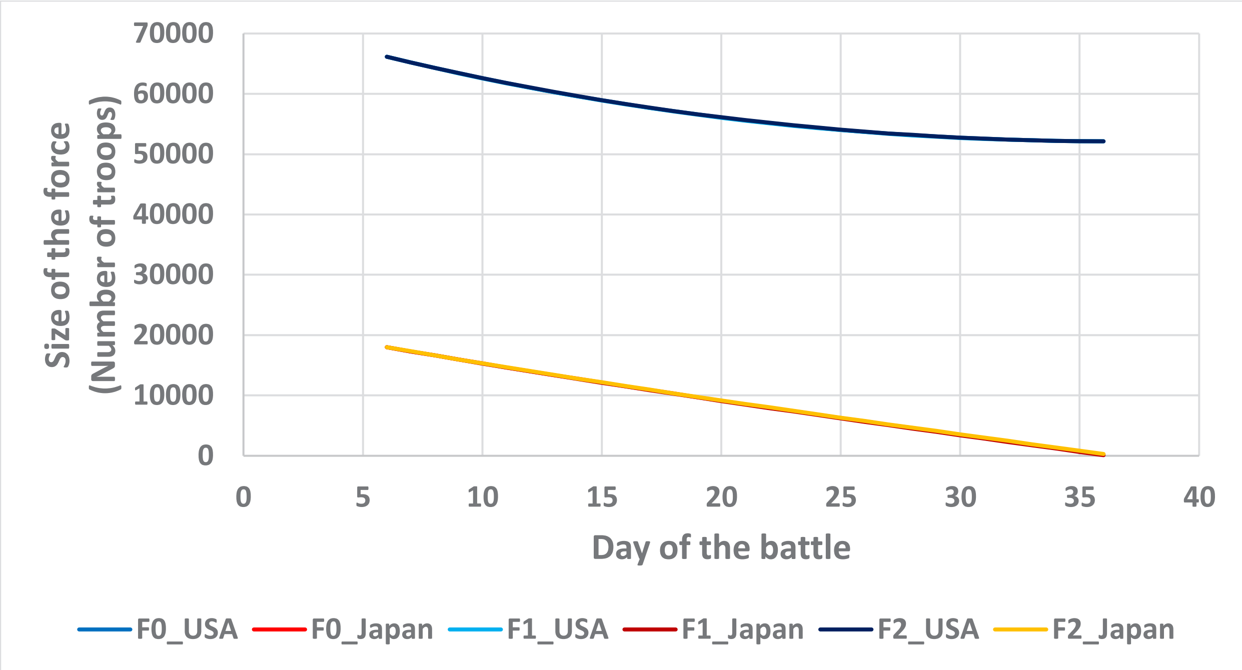

Figure 6. Sizes of the forces from USA and from Japan, according to different assumptions concerning the size of the Japanese force at the end of the battle, Y_T.

A close inspection of the size of the Japanese force in Figure 7, shows that the different assumptions and cases give different results. Figure 8 gives a detailed description of how the derived size of the force from USA dynamically is influenced by the alternative assumptions concerning the terminal size of the Japanese force, Y_T.

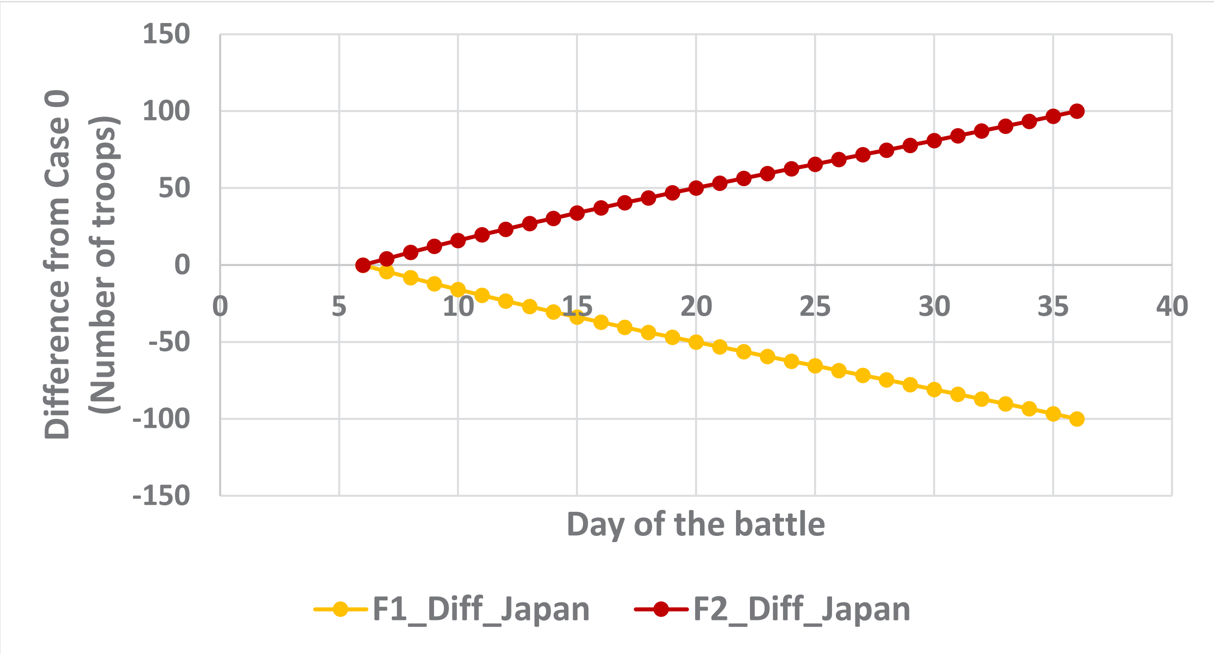

Figure 7. Differences of the size of the force from Japan from Case 0, according to different assumptions concerning the size of the Japanese force at the end of the battle, Y_T.

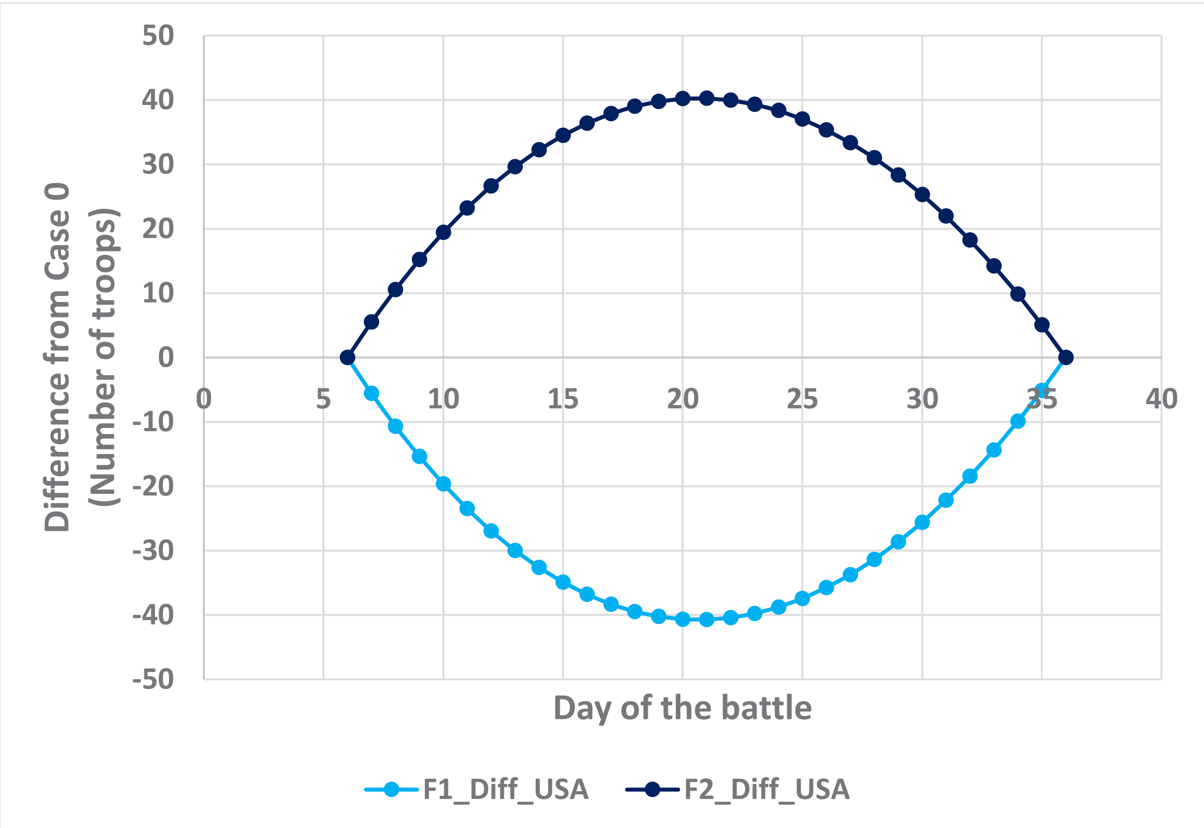

Figure 8. Differences of the size of the force from USA from Case 0, according to different assumptions concerning the size of the Japanese force at the end of the battle, Y_T.

The deviations between the true and estimated force sizes from USA, for the alternative cases, are reported in Figure 9. Note that these deviations are zero at t=0 and at t = T. The deviations can partly be explained by the variations of the force change that are described in Figure 3.

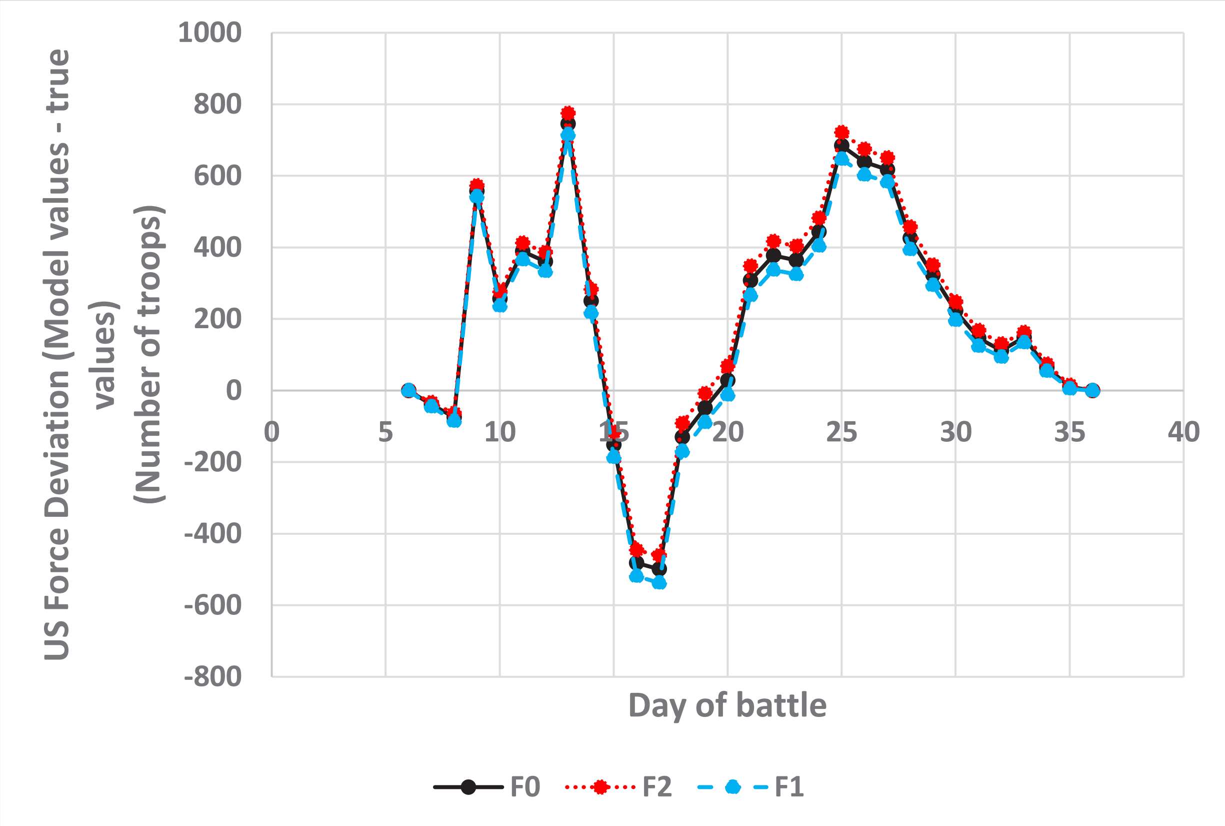

Figure 9. Deviations of the size of the force from USA from the calculated values, according to different assumptions concerning the size of the Japanese force at the end of the battle, Y_T.

In Table 5, we find that the different estimates of the attrition coefficients, a and b, are very similar in the different studies.

Table 5. Estimations of the attrition coefficient values.

	Estimated value of a	**Estimated value of b**	**R^2**
Lohmander Case 0	0.05347	0.01045	0.99997
Lohmander Case 1	0.05379	0.01051	0.99997
Lohmander Case 2	0.05315	0.01039	0.99997
Engel (1954)	0.0544	0.0106	0.9937
Braun (1993) via Engel (1954)	0.0544	0.0106	No information
Washburn and Kress (2009) via Engel (1954)	0.0544	0.0106	No information
Stymfal (2022)	0.0532	0.0105	0.9944

The new results may be summarized this way: To determine the attrition coefficients and the complete dynamics of the battle in continuous time, the following procedure is used: First, the general solution to the Lanchester differential equation system, which is a homogenous second order differential equation system, is derived. This represents a 2-dimensional Two Point Boundary Value Problem. Four parameters are determined, via a nonlinear simultaneous equation system with four equations. In these equations, the initial and terminal sizes of the two forces, are parameters. A 4-dimensional fix point iteration algorithm is developed and implemented as an included computer code, that rapidly solves the nonlinear equation system. After 40 iterations, the absolute relative errors in all equations are less than 10^{-12} . Then, a discrete time version of the Lanchester differential equation system, with stochastic attrition coefficients, is defined as a difference equation system. Stochastic variables are added to the attrition coefficients in different time periods, keeping the expected attrition coefficient constant. The effect of increasing risk in the attrition coefficients that determines how the time derivative of force X is affected by force Y at different points in time is analyzed. It is shown that the expected size of the force X is a strictly convex function of the risk in that attrition coefficient. Hence, according to the Jensen's inequality, the expected size of the force X is a strictly increasing function of the risk in that attrition coefficient. Comparative statics analysis shows that, in case the attrition coefficients in different periods are stochastic, and the system parameters are determined according to the suggested procedure, then the expected attrition coefficients obtain higher values than if the attrition coefficients would be constant over time. This can explain differences between attrition coefficient estimates based on different methods and coefficient risk assumptions.

4. Discussion

Wars and battles are phenomena with negative consequences of many kinds. The numbers of dead and injured soldiers are often large and the values of damaged infrastructure and lost property are often enormous. Destroyed environments and large numbers of dead and injured civilians, are typical and terrible consequences of wars. For these reasons, it is very important that we do our best to avoid battles and wars. In case they can not be completely avoided, we should at least manage and control them in the best possible ways, reducing unnecessary consequences. In order to optimize the response to an aggressor, it is important to know how different decisions are expected to influence the outcome of a war. In this situation, it is necessary to have good and reliable estimates of attrition coefficients in different kinds of situations.

In the case of Iwo Jima, the Japanese force had prepared the defense of the island, as described by the earlier studies. It is not surprising that the number of killed soldiers from USA, per soldier from Japan, per day, namely attrition coefficient a, is higher than the number of killed soldiers from Japan, per soldier from USA, per day, denoted attrition coefficient b. As we see in Table 5, this is also the case. The attrition coefficient a is approximately five times larger than the attrition coefficient b. However, even if the Japanese troops could kill many more soldiers from USA, per soldier and day, USA could still win the battle, since they had many more soldiers available when the battle started. This follows from the famous Lanchester square law, Lanchester (1916). This law is also very well described and demonstrated in Braun (1993).

Note that, the number of troops that USA sent to Iwo Jima, was sufficiently large to win the battle. However, it has not been proved that this was the optimal number of troops. With more US troops sent to the island, the number of killed and wounded US troops would probably be lower and the battle could have ended more rapidly. The optimal distribution of troops to different battle fields is an important optimization problem.

Some research questions, strongly motivating to consider for future projects, in order to reduce the negative consequences from possible attacks, are the following:

- How are the attrition coefficients influenced by the type of battle, different kinds of troops, arms, and weather conditions?
- How should different possible battles be integrated in a relevant analysis of a complete war?
- How should a defender most rationally organize the total defense, against possible aggressors, via optimal distribution of troops and arms to different possible battle fields?

Reliable defense solutions, derived via rational defense analyses, may led to a more peaceful world. Efficient and reliable estimations of attrition coefficients, as suggested in this paper, are the necessary first steps.

5. Conclusions

This study has shown the following:

It is possible to determine the attrition coefficients of a battle, if the initial and terminal sizes of the forces of the involved parties are known, and the general solution to the relevant differential equation system can be derived. This means that detailed statistical data tables, representing the time series of the sizes of the involved forces, are not necessary. This is an important conclusion since it is often very difficult, costly, dangerous and/or impossible to get access to detailed and reliable military statistical data, particularly during wars that have not yet ended. In the earlier mentioned articles on the battle of Iwo Jima, the authors of those articles used different statistical procedures and approximations to estimate the attrition coefficients. Now, with the new estimation procedure, based on a general differential equation system solution and a numerical iteration algorithm, it is possible to rapidly obtain almost identical estimates of the attrition parameters, a comparison of which may be made from Table 5.

Furthermore, with the new procedure, it is also possible to instantly, in less than a second, determine how possible changes of different parameters, such as the not exactly known terminal size of the Japanese force, influence the estimated attrition parameters. This is reported in Table 5. The new procedure automatically reports not only the estimated attrition coefficients, but also the equations that describe the dynamics of the involved forces, as explicit functions of time.

If the attrition coefficients change over time, and the expected attrition coefficients are estimated via regression analysis based on the complete detailed time series of the involved forces, then the expected attrition coefficients are larger, than if the attrition coefficient estimates are calculated with the method developed in this paper, or according to the earlier methods mentioned in this paper. The reason is that the expected size of a force at a later point in time is a strictly increasing function of the risk in the attrition parameters during earlier points in time. Hence, it is necessary that the expected attrition coefficient increases, if the size of the force should decrease to the same terminal size, as if the attrition coefficients would be constant over time.

Acknowledgements

The author is most grateful for the important comments and work of the reviewers and editors. Their efforts improved the final version of the manuscript.

References

Bracken, J. (1995). Lanchester models of Ardennes Campaign. *Naval Research Logistics,* 42(1), 559–577. https://onlinelibrary.wiley.com/doi/10.1002/1520-6750(199506)42:4%3C559::AID-NAV3220420405%3E3.0.CO;2-R

Braun, M. (1993). Differential Equations and Their Applications. *Springer-Verlag*, New York, 4 ed. https://link.springer.com/book/10.1007/978-1-4612-4360-1

Engel, J.H. (1954). A Verification of Lanchester's Law. *Journal of the Operations Research Society of America,* 2(2):163-171. https://pubsonline.informs.org/doi/pdf/10.1287/opre.2.2.163

Hung, C.Y., Yang, G. and Deng, P. *et al.* (2005). Fitting Lanchester's square law to the Ardennes Campaign. *Journal of the Operations Research Society of America,* **56**, 942–946. https://doi.org/10.1057/palgrave.jors.2601878

Iannelli, M. and Pugliese, A. (2014). Competition among species. In: An Introduction to Mathematical Population Dynamics, vol 79. *Springer*, Cham. https://doi.org/10.1007/978-3-319-03026-5_7

Jensen, J. L.W. V. (1906). Sur les fonctions convexes et les inégalités entre les valeurs moyennes. *Acta Math.* 30 175 - 193. https://doi.org/10.1007/BF02418571

Lanchester F.W. (1916). Aircraft in Warfare: The Dawn of the Fourth Arm. *Constable*, London. https://ia804709.us.archive.org/23/items/aircraftinwarfar00lancrich/aircraftinwarfar00lancrich.pdf

Lohmander, P. (1986). Continuous extraction under risk. *IIASA, International Institute for Applied Systems Analysis*, Systems and Decisions Sciences, WP-86-16, March 1986. https://core.ac.uk/download/pdf/33894314.pdf

Lohmander, P. (1988). Continuous extraction under risk. *Systems Analysis – Modelling – Simulation*, Vol. 5, No. 2, 131-151. http://www.Lohmander.com/PL_SAMS_5_2_1988.pdf

Lohmander, P. (2019a). Four central military decision problems, General methods and solutions. *The Royal Swedish Academy of War Sciences Proceedings and Journal*, 2, 119-134. http://www.lohmander.com/PLRSAWS_19.pdf

Lohmander, P. (2019b). Optimal decisions and expected values in two player zero sum games with diagonal game matrixes—Explicit functions, general proofs and effects of parameter estimation errors. *International Robotics and Automation Journal*, *5(1)*, 186–198. https://medcraveonline.com/IRATJ/IRATJ-05-00193.pdf

Lohmander, P. (2023). Optimal Dynamic Control of Proxy War Arms Support, *Automation* 4(1), 31-56. https://doi.org/10.3390/automation4010004

Rothschild, M. and Stiglitz, J.E. (1970). Increasing risk: I. A definition. *Journal of Economic Theory*, Volume 2(3), 225-243, ISSN 0022-0531, https://doi.org/10.1016/0022-0531(70)90038-4, https://www.sciencedirect.com/science/article/pii/0022053170900384

Rothschild, M. and Stiglitz, J.E. (1971). Increasing risk II: Its economic consequences. *Journal of Economic Theory*, 3(1), 66-84, ISSN 0022-0531, https://doi.org/10.1016/0022-0531(71)90034-2, https://www.sciencedirect.com/science/article/pii/0022053171900342

Shatz, H.J. (2020). Economic Competition in the 21st Century. RAND Corporation, Santa Monica, California. https://www.rand.org/pubs/research_reports/RR4188.html

Stymfal, M.G. (2022). Revisiting Engel's verification of Lanchester's square law using battle of Iwo Jima data. Thesis, *Naval Postgraduate School*, NPS, Monterey, California, USA. https://apps.dtic.mil/sti/trecms/pdf/AD1201776.pdf

Tam, J.H. (1998). Application of Lanchester combat model in the Ardennes campaign. *Natural Resource Modeling*, 11, 95-116. https://doi.org/10.1111/j.1939-7445.1998.tb00302.x

Washburn, A. and Kress, M. (2009). Combat Modeling, *Springer Science*, New York. https://archive.org/details/springer_10.1007-978-1-4419-0790-5

APPENDIX

A.1 Numerical Appendix

The following software, Iter_diff_230827_1300.bas, was developed for QB64.exe. It contains the iteration algorithm applied in this paper. Note that the parameters may be changed in different cases.

```
Rem
Rem Iter_diff
Rem Peter Lohmander 230827
Screen _NewImage(1800, 2000, 256)
Open "C:\Users\Peter\OneDrive\Desktop\Iter\IterOut.txt" For Output As #2
DefDbl A-Z

m1_0 = 1
m2_0 = 1
a_0 = 0.02
```

```
r_0 = 0.02
b_0 = r_0 * r_0 / a_0
m1 = m1_0
m2 = m2_0
a = a_0
r = r_0
b = b_0
T = 30
x0 = 66150
xT = 52135
y0 = 18000
yT = 200
h = 0.3
Cls
Print ""
Print "  Relative errors:"
Print ""
Print "               x0Err         y0Err         xTErr         yTErr"
Print #2, ""
Print #2, "  Relative errors:"
Print #2, ""
Print #2, "               x0Err         y0Err         xTErr         yTErr"

For i = 1 To 40
   m1 = x0 - m2
   m2 = xT * Exp(-r * T) - m1 * Exp(-2 * r * T)
   a = r * (m1 - m2) / y0
   k = (Exp(-r * T) * (r / a) * m1 - yT) / ((r / a) * m2)
   If k > 1 Then r = r + h * (Log(k) / T - r)
   b = r * r / a

   x0Est = m1 + m2
   y0Est = r / a * m1 - r / a * m2
   xTEst = Exp(-r * T) * m1 + Exp(r * T) * m2
   yTEst = Exp(-r * T) * r / a * m1 - Exp(r * T) * r / a * m2
   x0Err = (x0Est - x0) / x0
   y0Err = (y0Est - y0) / y0
   xTErr = (xTEst - xT) / xT
   yTErr = (yTEst - yT) / yT

   Print "      ";
   Print Using "###.############"; x0Err; y0Err; xTErr; yTErr
   Print #2, "      ";
   Print #2, Using "###.############"; x0Err; y0Err; xTErr; yTErr
Next i

Print ""
Print "  Initial and terminal conditions:"
Print ""
Print "      x0 = "; x0
Print "      y0 = "; y0
Print "      xT = "; xT
```

```
Print "      yT = "; yT
Print ""
Print "  Other parameters:"
Print ""
Print "      T = "; T
Print "      h = "; h
Print ""
Print "  Initial values of estimated parameters:"
Print ""
Print "      a_0  = "; a_0
Print "      b_0  = "; b_0
Print "      r_0  = "; r_0
Print "      m1_0 = "; m1_0
Print "      m2_0 = "; m2_0
Print ""
Print "  Estimated parameter values: "
Print ""
Print "      a  = "; a
Print "      b  = "; b
Print "      r  = "; r
Print "      m1 = "; m1
Print "      m2 = "; m2

Print ""
Print "  Estimated equations:"
Print ""
Print "      x(t) = ";
Print Using "######.###"; m1;
Print " * EXP(";
Print Using "##.########"; -r;
Print " * t )  + ";
Print Using "######.###"; m2;
Print " * EXP(";
Print Using "##.########"; r;
Print " * t )"
Print ""
Print "      y(t) = ";
Print Using "######.###"; r / a * m1;
Print " * EXP(";
Print Using "##.########"; -r;
Print " * t )  - ";
Print Using "######.###"; r / a * m2;
Print " * EXP(";
Print Using "##.########"; r;
Print " * t )"
Print #2, ""
Print #2, "  Initial and terminal conditions:"
Print #2, ""
Print #2, "      x0 = "; x0
Print #2, "      y0 = "; y0
Print #2, "      xT = "; xT
Print #2, "      yT = "; yT
```

```
Print #2, ""
Print #2, "  Other parameters:"
Print #2, ""
Print #2, "      T = "; T
Print #2, "      h = "; h
Print #2, ""
Print #2, "  Initial values of estimated parameters:"
Print #2, ""
Print #2, "      a_0  = "; a_0
Print #2, "      b_0  = "; b_0
Print #2, "      r_0  = "; r_0
Print #2, "      m1_0 = "; m1_0
Print #2, "      m2_0 = "; m2_0
Print #2, ""
Print #2, "  Estimated parameter values: "
Print #2, ""
Print #2, "      a  = "; a
Print #2, "      b  = "; b
Print #2, "      r  = "; r
Print #2, "      m1 = "; m1
Print #2, "      m2 = "; m2
Print #2, ""
Print #2, "  Estimated equations:"
Print #2, ""
Print #2, "      x(t) = ";
Print #2, Using "######.###"; m1;
Print #2, " * EXP(";
Print #2, Using "##.########"; -r;
Print #2, " * t )  + ";
Print #2, Using "######.###"; m2;
Print #2, " * EXP(";
Print #2, Using "##.########"; r;
Print #2, " * t )"
Print #2, ""
Print #2, "      y(t) = ";
Print #2, Using "######.###"; r / a * m1;
Print #2, " * EXP(";
Print #2, Using "##.########"; -r;
Print #2, " * t )  - ";
Print #2, Using "######.###"; r / a * m2;
Print #2, " * EXP(";
Print #2, Using "##.########"; r;
Print #2, " * t )"

Close #2
End
```

A.2 Empirical Appendix

Day of the battle	Size of the force from USA during the battle of Iwo Jima (from Stymfal (2022)).
0	54000
1	52839
2	50945
3	56026
4	54885
5	53744
6	66150
7	65245
8	64373
9	62869
10	62334
11	61400
12	60662
13	59544
14	59340
15	59076
16	58774
17	58191
18	57254
19	56636
20	56055
21	55303
22	54791
23	54393
24	53933
25	53342
26	53067
27	52799
28	52730
29	52603
30	52502
31	52407
32	52299
33	52150
34	52150
35	52150
36	52135